

# Casimir-Lifshitz force out of thermal equilibrium and heat transfer between arbitrary bodies

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**Abstract.** – We study the Casimir-Lifshitz force and the radiative heat transfer occurring between two arbitrary bodies, each one held at a given temperature, surrounded by environmental radiation at a third temperature. The system, in stationary configuration out of thermal equilibrium, is characterized by a force and a heat transfer depending on the three temperatures, and explicitly expressed in terms of the scattering operators of each body. We find a closed-form analytic expression valid for bodies of any geometry and dielectric properties. As an example, the force between two parallel slabs of finite thickness is calculated, showing the importance of the environmental temperature as well as the occurrence of a repulsive interaction. An analytic expression is also provided for the force acting on an atom in front of a slab. Our predictions can be relevant for experimental and technological purposes.

**Introduction.** – The quantum and thermal fluctuations of the electromagnetic field result in a force between any couple of polarizable bodies. This dispersion effect, originally predicted by Casimir for two parallel perfectly conducting plates at zero temperature, and later generalized by Lifshitz and coworkers to real dielectric materials at finite temperatures [1], becomes relevant for bodies separated by less than few microns. Today, the Casimir-Lifshitz force plays a major role in all fundamental and technological issues occurring at these separations, becoming widely interesting, from biological systems to micro-electromechanics [2,3].

The force shows two components, related to the purely quantum and thermal field fluctuations, respectively. The former dominate at short distances, and have been measured in several configurations since early times. The latter are relevant at large separations, with a weak total effect measured only recently [4,5]. The Casimir-Lifshitz force, largely studied for systems at thermal equilibrium, has been considered also for systems out of thermal equilibrium [6,7], as done for the atom-atom, plane-plane and atom-surface force in [8–10]. More recently, other particular configurations have been studied out of thermal equilibrium, involving infinitely thick planar and non planar

bodies, and atoms [11–15]. Indeed, non-equilibrium effects received a renewed interest since the recent discovery that systems driven out of thermal equilibrium may show new qualitative and quantitative behaviors, namely the possibility of a repulsive force, and of a strong force tunability [16]. Such peculiar characteristics allowed a non-equilibrium system to be used for the first measurement of thermal effects, by exploiting trapped ultracold atomic gases [5].

Such a force, related to the correlations of the electromagnetic field, shares a common formalism with radiative heat transfer [17], recently object of several investigation for both fundamental and technological issues [18].

Due to a less direct formalism, calculations of forces and heat transfer for systems out of equilibrium have been performed only for some ideal configurations (atoms at zero temperature, infinitely extended surfaces, . . .), and a general theory able to take into account arbitrary materials and geometries at different temperatures in an arbitrary thermal environment, is still missing. In this paper we derive such a theory, and we obtain a closed-form explicit expression for the most general problem of the force and heat transfer between two arbitrary bodies at two different temperatures placed in a thermal environment having

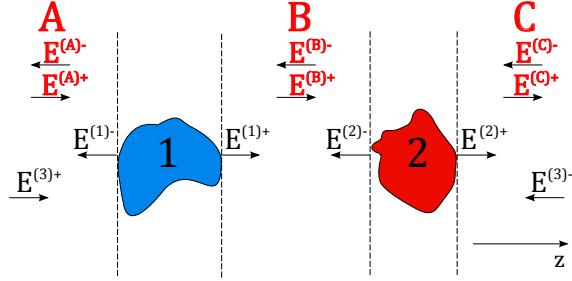


Fig. 1: (color online) Scheme of the system. The body 1 (2) is held at temperature  $T_1$  ( $T_2$ ), while the environment has temperature  $T_3$ .

a third (in general different) temperature. In addressing this problem we describe the two bodies by means of their scattering operators. This approach, successfully used to calculate the Casimir-Lifshitz force both at [19, 20] and out [13] of thermal equilibrium, even if formally equivalent to a Green function technique, presents the advantage of requiring only single-body operators rather than the complete solution for the composite system.

**The physical system.** – The system is made of two bodies, labeled with indexes 1 and 2, which we assume separated by at least an infinite plane. This assumption, introduced to exclude two concatenated bodies, is not necessary in approaches based on scattering theory [20], nonetheless it makes the formalism much easier to follow and it is verified practically in all the common experimental configurations (e.g. plane-plane, sphere-plane, atom-surface). We are going to calculate the component of the forces acting on the two bodies along the axis perpendicular to such a plane, to which we refer as the  $z$  axis, as well as the heat flux on each of them. A scheme of the system is shown in figure 1, where three distinct regions A, B and C are defined. The body 1 (2) is assumed to be at a *local* thermal equilibrium at temperature  $T_1$  ( $T_2$ ), and on the two bodies impinges a thermal radiation at the third temperature  $T_3$ . One can imagine a system where the two bodies are inside, and far from the surfaces, of a much bigger cell (the reservoir), held locally at thermal equilibrium at temperature  $T_3$ . The three temperatures remain constant in time, so that the system assumes a stationary regime. If at least one of the two bodies is microscopic, the validity of his hypothesis requires a more careful analysis: generally speaking it remains acceptable for time scales much shorter than the typical time of evolution of the microscopic body.

The force  $\mathbf{F}$  acting on any of the two bodies and the heat absorbed by it per unit of time can be evaluated calculating the fluxes

$$\mathbf{F} = \int_{\Sigma} \langle \mathbb{T}(\mathbf{R}, t) \rangle_{\text{sym}} \cdot d\mathbf{\Sigma}, \quad H = - \int_{\Sigma} \langle \mathbf{S}(\mathbf{R}, t) \rangle_{\text{sym}} \cdot d\mathbf{\Sigma} \quad (1)$$

of the symmetrized averages  $\langle AB \rangle_{\text{sym}} = \langle AB + BA \rangle / 2$  of the Maxwell stress tensor  $\mathbb{T}$  and Poynting vector  $\mathbf{S}$  in vacuum

$$T_{ij}(\mathbf{R}, t) = \epsilon_0 \left[ E_i E_j + c^2 B_i B_j - \frac{1}{2} (E^2 + c^2 B^2) \delta_{ij} \right] \quad (2)$$

$$\mathbf{S}(\mathbf{R}, t) = \epsilon_0 c^2 \mathbf{E} \times \mathbf{B}$$

( $i, j = x, y, z$ ), through an arbitrary closed and oriented surface  $\Sigma$  enclosing the body. Choosing  $\Sigma$  as a parallelepiped having two of its faces orthogonal to the  $z$  axis (and on two opposite sides of a given body) and letting the surface of these sides tend to infinity, one finds that the  $m$  component ( $m = x, y, z$ )  $F_m$  of the force (the heat transfer) has non negligible contribution only from the difference of the fluxes of the  $\langle T_{mz} \rangle$  component of the tensor (the  $\langle S_z \rangle$  component of the Poynting vector) on the two sides of the considered body. For the electric and magnetic fields appearing in eq. (2) we use the following mode decomposition

$$\mathbf{E}(\mathbf{R}, t) = 2\text{Re} \left[ \int_0^{+\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \mathbf{E}(\mathbf{R}, \omega) \right] \quad (3)$$

where

$$\mathbf{E}(\mathbf{R}, \omega) = \sum_{p, \phi} \int \frac{d^2 \mathbf{k}}{(2\pi)^2} e^{i\mathbf{K}^{\phi} \cdot \mathbf{R}} \hat{\mathbf{e}}_p^{\phi}(\mathbf{k}, \omega) E_p^{\phi}(\mathbf{k}, \omega). \quad (4)$$

In this representation, known as angular spectrum representation [21, 22], a mode has amplitude  $E_p^{\phi}(\mathbf{k}, \omega)$ , corresponding to the frequency  $\omega$ , the transverse wavevector  $\mathbf{k} = (k_x, k_y)$ , the transverse polarization  $p$  taking the values 1 (TE) and 2 (TM), and the direction of propagation along the  $z$  axis  $\phi = \pm 1$  (with shorthand notation  $\phi = \pm$  in the polarization vectors and field amplitudes). In this approach, which proves to be convenient in our planar-like geometry,  $k_z = \sqrt{\omega^2/c^2 - \mathbf{k}^2}$  is a dependent variable, and the three-dimensional wavevector is noted as  $\mathbf{K}^{\phi} = (\mathbf{k}, \phi k_z)$ . We have also introduced the polarization unit vectors, defined as  $\hat{\mathbf{e}}_{\text{TE}}^{\phi}(\mathbf{k}, \omega) = \hat{\mathbf{z}} \times \hat{\mathbf{k}}$  and  $\hat{\mathbf{e}}_{\text{TM}}^{\phi}(\mathbf{k}, \omega) = \hat{\mathbf{e}}_{\text{TE}}^{\phi}(\mathbf{k}, \omega) \times \hat{\mathbf{A}}^{\phi}$  where  $\hat{\mathbf{z}} = (0, 0, 1)$  and  $\hat{\mathbf{A}} = \mathbf{A}/A$ . The analogous expression for the magnetic field can be directly deduced from Maxwell's equations. The field  $E^{(\gamma)\phi}$  in fig. 1 is the part of the field emitted by the body  $\gamma$  (for  $\gamma = 1, 2$ ), of the environmental field (for  $\gamma = 3$ ), or of the *total* field in each region (for  $\gamma = A, B, C$ ) propagating in direction  $\phi$ . The fluxes of  $\langle T_{mz} \rangle$  and  $\langle S_z \rangle$  through a given plane  $z = \bar{z}$  can now be explicitly expressed in the following way

$$\Phi_m(\bar{z}) = \int_{z=\bar{z}} d^2 \mathbf{r} \langle T_{mz} \rangle_{\text{sym}} \quad (5)$$

$$= - \sum_p \int \frac{d^2 \mathbf{k}}{(2\pi)^2} \left( \sum_{\phi=\phi'} \int_{ck}^{+\infty} \frac{d\omega}{2\pi} + \sum_{\phi \neq \phi'} \int_0^{ck} \frac{d\omega}{2\pi} \right)$$

$$\times \frac{2\epsilon_0 c^2 k_z}{\omega^2} \langle p, \mathbf{k} | C^{(\gamma)\phi\phi'} | p, \mathbf{k} \rangle \times \begin{cases} \phi k_m & m = x, y \\ k_z & m = z \end{cases}$$

$$\begin{aligned}\varphi(\bar{z}) &= \int_{z=\bar{z}} d^2\mathbf{r} \langle S_z \rangle_{\text{sym}} \\ &= \sum_p \int \frac{d^2\mathbf{k}}{(2\pi)^2} \left( \sum_{\phi=\phi'} \int_{ck}^{+\infty} \frac{d\omega}{2\pi} + \sum_{\phi \neq \phi'} \int_0^{ck} \frac{d\omega}{2\pi} \right) \quad (6) \\ &\times \frac{2\epsilon_0 c^2 \phi k_z}{\omega} \langle p, \mathbf{k} | C^{(\gamma)\phi\phi'} | p, \mathbf{k} \rangle\end{aligned}$$

where  $\gamma$  denotes the region in which  $\bar{z}$  is located. Each of these contributions do not depend on the coordinate of the plane chosen to calculate the flux, provided that this surface is located inside a given region  $A$ ,  $B$  or  $C$ . These quantities are expressed as a function of the matrix  $C^{(\gamma)\phi\phi'}$

$$\begin{aligned}\langle E_p^{(\gamma)\phi}(\mathbf{k}, \omega) E_{p'}^{(\gamma)\phi'\dagger}(\mathbf{k}', \omega') \rangle_{\text{sym}} \\ = \frac{1}{2} \langle E_p^{(\gamma)\phi}(\mathbf{k}, \omega) E_{p'}^{(\gamma)\phi'\dagger}(\mathbf{k}', \omega') + E_{p'}^{(\gamma)\phi'\dagger}(\mathbf{k}', \omega') E_p^{(\gamma)\phi}(\mathbf{k}, \omega) \rangle \\ = 2\pi\delta(\omega - \omega') \langle p, \mathbf{k} | C^{(\gamma)\phi\phi'} | p', \mathbf{k}' \rangle\end{aligned} \quad (7)$$

being the correlating function between two amplitudes of the *total* field propagating in directions  $\phi$  and  $\phi'$ , and associated to a couple of modes  $(\mathbf{k}, p)$  and  $(\mathbf{k}', p')$ . We have explicitly inserted the conservation of frequency, direct consequence of the time invariance (moving bodies are excluded from our calculation) characterizing our system. In eq. (7) the correlators are written as matrix elements of a matrix  $C^{\phi\phi'}$ : in our notation, the matrices are defined on the space  $(p, \mathbf{k})$  being  $p = 1, 2$  and  $\mathbf{k} \in \mathbb{R}^2$  and thus the product of two matrices  $A$  and  $B$  is given by

$$\begin{aligned}\langle p, \mathbf{k} | \mathcal{A}\mathcal{B} | p', \mathbf{k}' \rangle = \sum_{p''} \int \frac{d^2\mathbf{k}''}{(2\pi)^2} \langle p, \mathbf{k} | \mathcal{A} | p'', \mathbf{k}'' \rangle \\ \times \langle p'', \mathbf{k}'' | \mathcal{B} | p', \mathbf{k}' \rangle.\end{aligned} \quad (8)$$

**At equilibrium.** – For a system at thermal equilibrium, the correlators are directly given by the fluctuation-dissipation theorem

$$\begin{aligned}\langle E_i(\mathbf{R}, \omega) E_j^*(\mathbf{R}', \omega') \rangle_{\text{sym}} = 2\pi\delta(\omega - \omega') \frac{2}{\omega} N(\omega, T) \\ \times \text{Im}G_{ij}(\mathbf{R}, \mathbf{R}', \omega),\end{aligned} \quad (9)$$

where the purely quantum and the thermal fluctuations are clearly distinguishable in the factor  $N(\omega, T) = \hbar\omega[1/2 + n(\omega, T)]$ , with  $n(\omega, T) = (\exp[\hbar\omega/(k_B T)] - 1)^{-1}$ , and  $G_{ij}$  are the components of the Green function  $\bar{\mathbf{G}}$  associated to the system of the two bodies, solution of the differential equation

$$\left[ \nabla_{\mathbf{R}} \times \nabla_{\mathbf{R}} - \frac{\omega^2}{c^2} \epsilon(\omega, \mathbf{R}) \right] \bar{\mathbf{G}}(\mathbf{R}, \mathbf{R}', \omega) = \frac{\omega^2}{\epsilon_0 c^2} \bar{\mathbf{I}} \delta(\mathbf{R} - \mathbf{R}') \quad (10)$$

being  $\bar{\mathbf{I}}$  the identity dyad and  $\epsilon(\omega, \mathbf{R})$  the dielectric function of the system. By expressing the Green function in terms of the reflection scattering matrices operators  $\mathcal{R}$  of

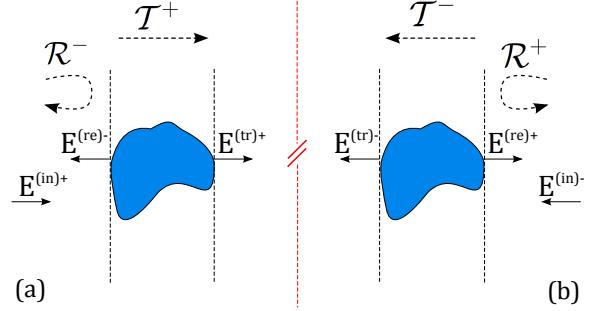


Fig. 2: Schemes for scattering operators.

the two bodies, one recovers the force acting on the body 1 along the  $z$  axis [19, 20]:

$$\begin{aligned}F^{(\text{eq})}(T) = -2\text{ReTr} \left\{ \frac{k_z}{\omega} N(\omega, T) \right. \\ \times \left. \left[ U^{(12)} \mathcal{R}^{(1)+} \mathcal{R}^{(2)-} + U^{(21)} \mathcal{R}^{(2)-} \mathcal{R}^{(1)+} \right] \right\},\end{aligned} \quad (11)$$

where  $U^{(12)} = (1 - \mathcal{R}^{(1)+} \mathcal{R}^{(2)-})^{-1}$  and  $U^{(21)} = (1 - \mathcal{R}^{(2)-} \mathcal{R}^{(1)+})^{-1}$ . The definitions of the trace and the reflection operators in (11) will become explicit in the following.

**Out of equilibrium.** – For a system out of thermal equilibrium, the theorem (9) is not valid, and the expression of the correlators is not explicit in general. Nevertheless, in the particular case of stationary non-equilibrium, this is possible by *tracing back* the knowledge of the correlators to the description of the fields emitted by each body alone and by the environment: this will be done by defining the operators describing the scattering produced by the presence of each body. To this end, taking into account a single body (say object 1), we consider an incoming field coming from the left side (region A in figure 1), as shown in figure 2(a): this field produces, in general, a field on both sides of the body.

We call the field produced in region A *reflected* and the one in region B *transmitted*, and we introduce the operators  $\mathcal{R}^-$  and  $\mathcal{T}^+$  relating the amplitudes of these outgoing fields to the amplitudes of the incoming field  $E_p^{(\text{in})+}(\mathbf{k}, \omega)$ . Gathering in a vectorial notation all the incident, reflected and transmitted modes (for any  $p$ ,  $\mathbf{k}$  and  $\omega$ )  $E^{(\text{in})+}$ ,  $E^{(\text{re})-}$  and  $E^{(\text{tr})+}$  respectively, the definitions of the scattering operators read  $E^{(\text{re})-} = \mathcal{R}^- E^{(\text{in})+}$  and  $E^{(\text{tr})+} = \mathcal{T}^+ E^{(\text{in})+}$ . An analogous procedure defines  $\mathcal{R}^+$  and  $\mathcal{T}^-$  (see figure 2(b)). If a body is at rest, the scattering process conserves the frequency and thus the matrix element  $\langle p, \mathbf{k}, \omega | \mathcal{S} | p', \mathbf{k}', \omega' \rangle$  of any scattering operator  $\mathcal{S}$  is proportional to  $2\pi\delta(\omega - \omega')$ . For convenience, we will work from now on with scattering operators  $\mathcal{S}(\omega)$  at a given frequency  $\omega$  and defined in the subspace  $(p, \mathbf{k})$ . Before going further, it is useful to introduce a *modified* transmission operator  $\tilde{\mathcal{T}}^\phi$  which gives only the scattered part of the transmitted field. It is defined by the relation  $\mathcal{T}^\phi = 1 + \tilde{\mathcal{T}}^\phi$  and, differently from the ordinary  $\mathcal{T}^\phi$ , it goes to zero in the limit of absence of the body.

To build the correlators, we need the expression of the *total field* in each region  $\gamma = A, B, C$  of figure 1, which originates from the fields emitted by the bodies and the environment. We would like to express the total field in each region through the scattering operators. This can be simply obtained by summing up all the possible multiple-scattering events: for example the fields propagating in the two directions in region B are the solutions of

$$\begin{cases} E^{(B)+} = E^{(1)+} + \mathcal{T}^{(1)+} E^{(3)+} + \mathcal{R}^{(1)+} E^{(B)-} \\ E^{(B)-} = E^{(2)-} + \mathcal{T}^{(2)-} E^{(3)-} + \mathcal{R}^{(2)-} E^{(B)+} \end{cases} \quad (12)$$

The solutions of these equations as well as the analogous relations for regions A and C are straightforward and will not be given here explicitly.

We are then left with the calculation of the correlators of the fields produced by the bodies and the environment. As far as the environment is concerned, it corresponds to a free bosonic field at temperature  $T_3$  having correlators

$$\begin{aligned} \langle E_p^{(3)\phi}(\mathbf{k}, \omega) E_{p'}^{(3)\phi'\dagger}(\mathbf{k}', \omega') \rangle_{\text{sym}} &= \delta_{\phi, \phi'} \frac{\omega}{2\epsilon_0 c^2} N(\omega, T) \\ &\times \text{Re} \left( \frac{1}{k_z} \right) \delta_{pp'} (2\pi)^3 \delta(\omega - \omega') \delta(\mathbf{k} - \mathbf{k}'). \end{aligned} \quad (13)$$

As for the bodies 1 and 2, our assumption that a local temperature can be defined for each one, and remains constant in time, reasonably leads to assume that the part of the total field *emitted* by each body is the same it would be if the body was at thermal equilibrium with the environment at its own temperature. This hypothesis, already used in [5, 11–13, 16, 17], implies that the correlators of the field *emitted* by each body can still be obtained using the fluctuation-dissipation theorem (9) at its local temperature, where the Green function is now associated to each body in absence of the other one.

This procedure, together with the explicit connection between Green function and scattering operators (directly obtained from their definitions [23]) allows us to obtain an explicit expression of the correlators of the field emitted by each body  $i = 1, 2$ : in particular, for modes propagating in the same direction ( $\phi = \phi'$ ) we obtain

$$\begin{aligned} \langle E_p^{(i)\phi}(\mathbf{k}, \omega) E_{p'}^{(i)\phi\dagger}(\mathbf{k}', \omega') \rangle_{\text{sym}} &= \frac{\omega}{2\epsilon_0 c^2} N(\omega, T_i) 2\pi \delta(\omega - \omega') \\ &\times \langle p, \mathbf{k} | \left( \mathcal{P}_{-1}^{(\text{pw})} - \mathcal{R}^{(i)\phi} \mathcal{P}_{-1}^{(\text{pw})} \mathcal{R}^{(i)\phi\dagger} - \mathcal{T}^{(i)\phi} \mathcal{P}_{-1}^{(\text{pw})} \mathcal{T}^{(i)\phi\dagger} \right. \\ &\left. + \mathcal{R}^{(i)\phi} \mathcal{P}_{-1}^{(\text{ew})} - \mathcal{P}_{-1}^{(\text{ew})} \mathcal{R}^{(i)\phi\dagger} \right) | p', \mathbf{k}' \rangle \end{aligned} \quad (14)$$

while for  $\phi \neq \phi'$  we obtain

$$\begin{aligned} \langle E_p^{(i)\phi}(\mathbf{k}, \omega) E_{p'}^{(i)\phi'\dagger}(\mathbf{k}', \omega') \rangle_{\text{sym}} &= \frac{\omega}{2\epsilon_0 c^2} N(\omega, T_i) 2\pi \delta(\omega - \omega') \\ &\times \langle p, \mathbf{k} | \left( -\mathcal{R}^{(i)\phi} \mathcal{P}_{-1}^{(\text{pw})} \mathcal{T}^{(i)\phi'\dagger} - \mathcal{T}^{(i)\phi} \mathcal{P}_{-1}^{(\text{pw})} \mathcal{R}^{(i)\phi\dagger} \right. \\ &\left. + \mathcal{T}^{(i)\phi} \mathcal{P}_{-1}^{(\text{ew})} - \mathcal{P}_{-1}^{(\text{ew})} \mathcal{T}^{(i)\phi'\dagger} \right) | p', \mathbf{k}' \rangle. \end{aligned} \quad (15)$$

In these expressions, which represent a crucial intermediate result, we have introduced the notation  $\mathcal{P}_m^{(\text{pw/ew})} = k_z^m \Pi^{(\text{pw/ew})}$  (this definition will be also used for  $m \neq 1$  in the following), where  $\Pi^{(\text{pw})}$  and  $\Pi^{(\text{ew})}$  are the projectors on the propagative and evanescent sector, respectively.

The knowledge of the correlators (13), (14) and (15) allows the evaluation of the flux of  $\langle T_{zz} \rangle$  and  $\langle S_z \rangle$  in each region. Finally, the differences of such fluxes (1) provides a closed-form analytic expression of the  $z$  component of the force and the heat transfer *relative to body 1* expressed in terms of the three temperatures  $T_1, T_2$ , and  $T_3$ , and of the scattering operators of body 1 and 2:

$$\begin{aligned} F(T_1, T_2, T_3) &= \frac{F^{(\text{eq})}(T_1) + F^{(\text{eq})}(T_2)}{2} + \Delta_2(T_1, T_2, T_3) \\ H(T_1, T_2, T_3) &= \Delta_1(T_1, T_2, T_3). \end{aligned} \quad (16)$$

In eq. (16) we present the final result for the force as a sum of two contributions, the first term being a thermal average of the force  $F^{(\text{eq})}(T)$  at thermal equilibrium given by eq. (11), at the temperatures of the two bodies  $T_1$  and  $T_2$  [11, 12]. The two terms  $\Delta_1$  and  $\Delta_2$  defined in eq. (16) can be collected as The force and heat transfer on object 2 can be obtained from (16) by changing the sign, and by interchanging indexes 1 and 2, as well as + and - in its explicit expression. The term (17) is purely a non-equilibrium contribution, obeying the equality  $\Delta_m(T, T, T) = 0$ . We remark that eq. (17) contains terms proportional to the transmission operators  $\mathcal{T}^{(1)\pm}$  and  $\mathcal{T}^{(2)-}$ , resulting from taking into account the finiteness of objects 1 and 2, which were absent in previous investigations concerning infinitely thick bodies [11–13]. It is worth stressing that eq. (17) provides a finite value of the force and the heat transfer for any finite body 1. This property is not evident at first sight since the expression (17) contains also divergent terms. For instance, this is the case of the first term in the second line, proportional to the operator  $\mathcal{P}_{m-1}^{(\text{pw})}$ . Since this operator is by definition diagonal in the  $(\mathbf{k}, p)$  basis, its trace is proportional to  $(2\pi)^2 \delta(\mathbf{0})$ , thus divergent. Moreover, we observe that this term diverges for any choice of the body 1, since it is formally independent on its scattering operators. Nevertheless, the quantity (17) remains finite due to peculiar cancellations of several divergent terms. In order to show that, we must pay attention to all the terms which, in analogy with  $\mathcal{P}_{m-1}^{(\text{pw})}$ , do not go to zero in absence of body 1. This is the case of the operators  $U^{(12)}$ ,  $U^{(21)}$  and  $T^{(1)\phi}$  ( $\phi = +, -$ ), having all the identity operator as a limiting value in absence of the body 1. By making use of the definition of the  $\tilde{\mathcal{T}}^\phi$  operator and of the properties  $U^{(12)} = 1 + \mathcal{R}^{(1)+} \mathcal{R}^{(2)-} U^{(12)}$  (and analogous for  $U^{(21)}$ ) it is straightforward to show that all the terms not going to zero in absence of body 1 perfectly cancel each other, and only contributions proportional to either  $\mathcal{R}^{(1)\phi}$  or  $\tilde{\mathcal{T}}^{(1)\phi}$  operators remain, leading indeed to a finite value of both the force and the heat transfer on any body 1.

$$\begin{aligned}
 \Delta_m(T_1, T_2, T_3) = & (-1)^{m+1} \hbar \text{Tr} \left\{ \omega^{2-m} \left[ \frac{n_{21}}{2} \left( A_m(\mathcal{R}^{(2)-}, \mathcal{R}^{(1)+}) - (-1)^m A_m(\mathcal{R}^{(1)+}, \mathcal{R}^{(2)-}) \right) + n_{13} \mathcal{P}_m^{(\text{pw})} \mathcal{R}^{(1)-} \mathcal{P}_{-1}^{(\text{pw})} \mathcal{R}^{(1)-\dagger} \right. \right. \\
 & + (-1)^m n_{m3} \mathcal{P}_{m-1}^{(\text{pw})} + n_{31} \left[ (-1)^m B_m(\mathcal{R}^{(1)+}, \mathcal{R}^{(2)-}, \mathcal{T}^{(1)+}) - \left( \mathcal{P}_m^{(\text{pw})} \mathcal{R}^{(1)-} \mathcal{P}_{-1}^{(\text{pw})} \mathcal{T}^{(1)+\dagger} U^{(12)\dagger} \mathcal{R}^{(2)-\dagger} \mathcal{T}^{(1)-\dagger} + \text{h.c.} \right) \right] \\
 & + \left[ n_{12} \left( \mathcal{P}_{-1}^{(\text{pw})} + \mathcal{R}^{(2)-} \mathcal{P}_{-1}^{(\text{ew})} - \mathcal{P}_{-1}^{(\text{ew})} \mathcal{R}^{(2)-\dagger} - \mathcal{R}^{(2)-} \mathcal{P}_{-1}^{(\text{pw})} \mathcal{R}^{(2)-\dagger} \right) + n_{23} \mathcal{T}^{(2)-} \mathcal{P}_{-1}^{(\text{pw})} \mathcal{T}^{(2)-\dagger} \right. \\
 & \left. \left. + n_{13} \mathcal{R}^{(2)-} \mathcal{T}^{(1)+} \mathcal{P}_{-1}^{(\text{pw})} \mathcal{T}^{(1)+\dagger} \mathcal{R}^{(2)-\dagger} \right] U^{(21)\dagger} \mathcal{T}^{(1)-\dagger} \mathcal{P}_m^{(\text{pw})} \mathcal{T}^{(1)-} U^{(21)} + n_{32} B_m(\mathcal{R}^{(2)-}, \mathcal{R}^{(1)+}, \mathcal{T}^{(2)-}) \right] \right\}. \tag{17}
 \end{aligned}$$

Equations (11) and (17) contain a trace, defined by the relation  $\text{Tr} \mathcal{A}(\omega) = \sum_p \int \frac{d^2 \mathbf{k}}{(2\pi)^2} \int_0^{+\infty} \frac{d\omega}{2\pi} \langle p, \mathbf{k} | \mathcal{A}(\omega) | p, \mathbf{k} \rangle$ , the function  $n_{ij} = n(\omega, T_i) - n(\omega, T_j)$  ( $i, j = 1, 2, 3$ ), and the two supplementary functions

$$\begin{aligned}
 A_m(\mathcal{R}^{(1)+}, \mathcal{R}^{(2)-}) = & U^{(12)} \left( \mathcal{P}_{-1}^{(\text{pw})} - \mathcal{R}^{(1)+} \mathcal{P}_{-1}^{(\text{pw})} \mathcal{R}^{(1)+\dagger} + \mathcal{R}^{(1)+} \mathcal{P}_{-1}^{(\text{ew})} - \mathcal{P}_{-1}^{(\text{ew})} \mathcal{R}^{(1)+\dagger} \right) \\
 & \times U^{(12)\dagger} \left( \mathcal{P}_m^{(\text{pw})} + (-1)^m \mathcal{R}^{(2)-\dagger} \mathcal{P}_m^{(\text{pw})} \mathcal{R}^{(2)-} + \mathcal{R}^{(2)-\dagger} \mathcal{P}_m^{(\text{ew})} + (-1)^m \mathcal{P}_m^{(\text{ew})} \mathcal{R}^{(2)-} \right), \tag{18} \\
 B_m(\mathcal{R}^{(1)+}, \mathcal{R}^{(2)-}, \mathcal{T}^{(1)+}) = & U^{(12)} \mathcal{T}^{(1)+} \mathcal{P}_{-1}^{(\text{pw})} \mathcal{T}^{(1)+\dagger} U^{(12)\dagger} \\
 & \times \left( \mathcal{P}_m^{(\text{pw})} + (-1)^m \mathcal{R}^{(2)-\dagger} \mathcal{P}_m^{(\text{pw})} \mathcal{R}^{(2)-} + \mathcal{R}^{(2)-\dagger} \mathcal{P}_m^{(\text{ew})} + (-1)^m \mathcal{P}_m^{(\text{ew})} \mathcal{R}^{(2)-} \right).
 \end{aligned}$$

**Some applications.** – We will now apply eq. (17) to two specific systems, namely atom-surface and slab-slab configurations. As anticipated before, although the derivation of the main result (17) benefits from the choice of the plane-wave basis, it can be applied to any couple of bodies (separated by a plane) even not satisfying translational invariance. An interesting candidate to check the validity of this feature is the atom-plane system. In this case, the scattering operators associated to the atom can be deduced by describing the atom as an induced dipole  $\mathbf{d}(\omega) = \alpha(\omega) \mathbf{E}(\mathbf{R}_A, \omega)$  proportional to the  $\omega$  component of the electric field calculated at the atomic position  $\mathbf{R}_A = (\mathbf{r}_A, z_A)$  through the atomic dynamical polarizability  $\alpha(\omega)$  (isotropy for the atomic polarizability has been assumed). Writing the field produced by the induced dipole and projecting it on the plane wave basis as done in [24] we obtain the following expression for the reflection and transmission atomic operators (for  $\phi = +, -$ )

$$\begin{aligned}
 \langle \mathbf{k}, p | \mathcal{R}_A^\phi(\omega) | \mathbf{k}', p' \rangle = & \frac{i\omega^2 \alpha(\omega)}{2\epsilon_0 c^2 k_z} \left( \hat{\epsilon}_p^\phi(\mathbf{k}, \omega) \cdot \hat{\epsilon}_{p'}^{-\phi}(\mathbf{k}', \omega) \right) \\
 & \times \exp[i(\mathbf{k}' - \mathbf{k}) \cdot \mathbf{r}_A] \exp[-i\phi(k_z + k'_z) z_A] \\
 \langle \mathbf{k}, p | \tilde{\mathcal{T}}_A^\phi(\omega) | \mathbf{k}', p' \rangle = & \frac{i\omega^2 \alpha(\omega)}{2\epsilon_0 c^2 k_z} \left( \hat{\epsilon}_p^\phi(\mathbf{k}, \omega) \cdot \hat{\epsilon}_{p'}^\phi(\mathbf{k}', \omega) \right) \\
 & \times \exp[i(\mathbf{k}' - \mathbf{k}) \cdot \mathbf{r}_A] \exp[-i\phi(k_z - k'_z) z_A]. \tag{19}
 \end{aligned}$$

These expressions clearly show that the wavevector  $\mathbf{k}$  and the polarization  $p$  are not conserved in the atomic scattering process, as a consequence of the absence of translational invariance. These operators can be inserted in eqs. (11) and (17) in order to obtain the equilibrium and non-equilibrium contributions to the force: to this aim, only the terms up to the first order in  $\mathcal{R}$  and  $\tilde{\mathcal{T}}$  are kept, coherently with a second-order perturbative approach with

respect to the electric charge (first order with respect to atomic polarizability). As shown for example in [24], this gives back the known result for the force at thermal equilibrium. As far as the non-equilibrium contribution is concerned, taking the atom on the left ( $z_A < 0$ ) of the surface  $z = 0$ , a simple calculation yields

$$\begin{aligned}
 \Delta_2(T_1, T_2, T_3) = & \frac{\hbar}{4\pi^2 \epsilon_0 c^2} \text{Im} \left\{ \sum_p \int_0^{+\infty} d\omega \omega^2 \alpha(\omega) \right. \\
 & \times \left[ n_{23} \int_0^{\frac{\omega}{c}} dk k (|\rho_p|^2 + |\tau_p|^2 - 1) \right. \\
 & + \int_0^{\frac{\omega}{c}} dk k (\hat{\epsilon}_p^+ \cdot \hat{\epsilon}_p^-) (n_{32} \rho_p e^{-2ik_z z_A} + n_{13} \rho_p^* e^{2ik_z z_A}) \\
 & \left. \left. + n_{12} \int_{\frac{\omega}{c}}^{+\infty} dk k (\hat{\epsilon}_p^+ \cdot \hat{\epsilon}_p^-) \rho_p^* e^{-2ik_z z_A} \right] \right\} \tag{20}
 \end{aligned}$$

where  $\rho_p$  and  $\tau_p$  are respectively the reflection and transmission Fresnel coefficients associated to a planar slab and the dependence of all the quantities inside the integral on  $\omega$  and  $k$  is implicit. We remark the fact that in eq. (20), the polarizability  $\alpha(\omega)$  is for an atom at temperature  $T_1$ . The first term in the square bracket in eq. (20) is a distance-independent contribution already discussed in [10, 16]. The second and the third terms are both distance-dependent: the former depends on the propagative sector, whereas for the latter only evanescent waves contribute. We have verified that, taking a ground-state atom ( $T_1 = 0$ ) and assuming that  $T_2$  and  $T_3$  are such that atomic excitation can be excluded (which amounts to replace  $\alpha(\omega)$  by its static value  $\alpha(0)$ ), we recover the result of [16], obtained using a different approach.

We consider now a second example for which we show quantitatively the new physical features produced by these

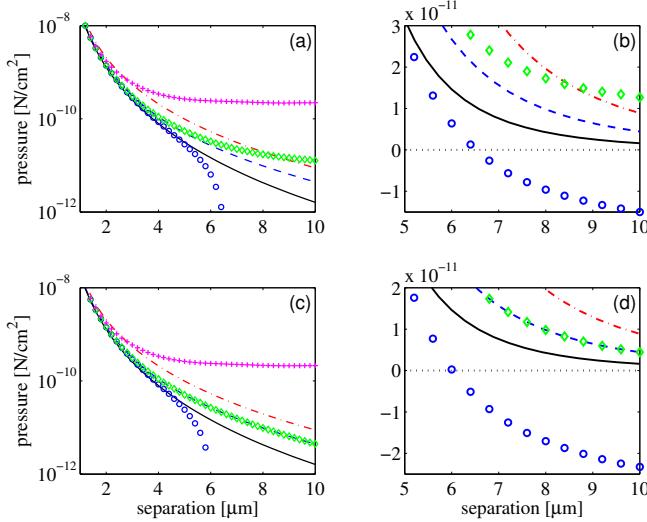


Fig. 3: (color online) Pressure acting on a  $2\mu\text{m}$  thick slab parallel to a  $1000\mu\text{m}$  thick slab (see text). Lines: equilibrium pressures at  $T = 0\text{ K}$  (solid),  $300\text{ K}$  (dashed),  $600\text{ K}$  (dash-dotted). Symbols: non-equilibrium pressures,  $T_3 = 0\text{ K}$  (circles),  $300\text{ K}$  (diamonds),  $600\text{ K}$  (crosses), with  $T_1 = 300\text{ K}$  and  $T_2 = 0\text{ K}$  in (a)-(b) and  $T_1 = T_2 = 300\text{ K}$  in (c)-(d).

terms. We calculate the non-equilibrium pressure on a  $2\mu\text{m}$  thick slab (body 1, fused silica) parallel to a  $1000\mu\text{m}$  thick slab (body 2, silicon). In figure 3(a)-(b) we show the case  $T_1 = 300\text{ K}$  and  $T_2 = 0\text{ K}$ , whereas in 3(c)-(d)  $T_1 = T_2 = 300\text{ K}$ : in both cases  $T_3$  takes the three values  $0\text{ K}$ ,  $300\text{ K}$  and  $600\text{ K}$ . The figure shows that in both cases (a)-(b) and (c)-(d) the non-equilibrium pressure can significantly differ from the equilibrium counterpart at any of the three temperatures involved. Moreover, both for equal and unequal  $T_1$  and  $T_2$ , the choice  $T_3 = 0$  produces a repulsive force starting around  $6\mu\text{m}$  of distance between the plates. This is particularly remarkable in the case  $T_1 = T_2 = 300\text{ K}$ , showing that the environmental temperature may play an important role for objects of finite thickness, qualitatively modifying the behavior of the force.

**Conclusions.** — We have derived a general expression for the Casimir-Lifshitz force and for the radiative heat transfer for systems out of thermal equilibrium, valid for bodies having arbitrary shape and dielectric function. Depending on the bodies and on the environmental temperatures, the force and heat transfer present several interesting degree of freedom. Due to its generality, our results allow a straightforward study of the force and heat transfer for systems involving bodies whose scattering matrices are analytically known (atoms, cylinders, spheres and slabs) [23, 25, 26], and also the investigation of most general bodies by a numerical evaluation of the scattering matrix. In particular, the heat transfer expression will allow to obtain more accurate estimations useful for past and future experiments as well as for technological appli-

cations such as solar cells. The force, which we calculated explicitly for an atom in front of a slab and numerically for two parallel slabs, can be significantly affected by thermal non-equilibrium with the environment, presenting transition from attractive to repulsive behaviors at distances of few microns.

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